Fluctuations and decoherence in classical chaos: A model study of a Kubo oscillator generated by a chaotic system

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We consider a Kubo oscillator whose stochasticity in frequency is generated by a 1.5-degree-of-freedom chaotic system. Based on the theory of multiplicative noise we show how fluctuation and decoherence and their relationship, which is analogous to the fluctuation-dissipation relation in many-body physics, can be realized in classical chaos. We numerically verify the basic theoretical propositions.

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I. INTRODUCTION

A key theme in nonlinear physics today is chaos in dynamical systems [1]. Recent experimental and theoretical research on few-degree-of-freedom systems has led to the discovery of new generic features in classical motion; regular and chaotic flows in phase space are, of course, the key notions. The chaotic motion is not associated with any random parameter or forces but is due to the unstable character of the trajectories in phase space.

The chaos in dynamical systems, although deterministic, is stochastic in nature in the statistical sense. It is therefore expected that statistical mechanical formalism [2–9] might be useful for description of classical chaos. For specific discrete systems such a formalism was considered by Kai and Tomita [2] and Oono and Takahashi [3]. Kadanoff and co-workers [4] have introduced a powerful method for characterizing multifractals based on certain partition functions. Widom *et al.* [5] discussed the example of Julia sets. The method of statistical mechanics was also followed by Kohmoto [6] to introduce entropy function and free energy function for multifractals which are a prerequisite for the existence of thermodynamics. These functions are related to Kolmogorov-Sinai entropy and Lyapunov exponents in the case of dynamical systems. For continuous low-dimensional chaotic systems, such as the Henon-Heiles Hamiltonian, the method of equilibrium statistical mechanics has recently been employed [7] to define temperature and entropy analogous to thermodynamics.

Apart from implementation of the methods of equilibrium statistical mechanics, kinetic description [9–14,16,17] has also been used over the years. Ever since the early numerical study of Chirikov mapping revealed that [10] the motion of a phase variable can be characterized by a simple random walk diffusion equation, attempts have been made to describe chaotic motion in terms of the Langevin or equivalently Fokker-Planck equation. It has now been realized that deterministic maps can result in long time diffusional processes and methods have been developed to predict successfully the corresponding diffusion coefficients [1]. While these studies are based on maps, identification of a noise term in the Lorenz equations after recasting it to an approximate Langevin form has been achieved by Nicolis and Nicolis [12] by successfully separating the distinct time scales involved in the dynamics using a center manifold method. Based on a strategy of separation of time scales Bianucci, West, and Grigolini [13] have considered a closed Hamiltonian system and shown that the system of interest following a slower dynamics obeys a Fokker-Planck equation having a canonical distribution which defines a temperature like quantity. Very recently we have proposed a general fluctuation-diffusion relation [14] (a Kubo relation in chaotic dynamics) for Hamiltonian systems which relates the largest Lyapunov exponent to the Fourier transform of the curvature-curvature (curvature of the potential) correlation function and shown that the theory of multiplicative noise can be a good natural description for classical chaos on several occasions.

In spite of a great deal of effort to derive the stochastic processes (for example, Brownian motion) from a purely deterministic dynamical model the problem has largely remained unsolved. From a theoretical standpoint it is worthwhile to distinguish two different situations. In the first case one is considered with the derivation of Brownian motion from a well-known system heat-bath model [15] comprised of $1+N$ linear oscillators (1 system oscillator and N bath oscillators. $N\rightarrow\infty$) In the second case [16,17] one is concerned with the derivation of a similar kind of stochastic process from a system–chaotic-bath model where the N-oscillator bath is replaced by a low-dimensional chaotic system. In the later context we specifically mention the work of Bianucci and co-workers [16,17] who, based on a simple two-dimensional (2D) map, have given a derivation of Fokker-Planck equation for the system of interest using only dynamical arguments where both the friction and diffusion terms are derived from the dynamical properties of the chaotic bath. In this paper we have addressed a related issue. We consider a Kubo oscillator, i.e., a harmonic oscillator with

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stochastic frequency where the stochasticity is generated by a 1.5-degree-of-freedom chaotic system which mimics a "thermal bath." Based on the theory of multiplicative noise [18] we analyze this system–chaotic-bath model to derive a stochastic process for the system oscillator for which damping of oscillation amplitude and diffusion coefficients are explicitly derived from the properties of the bath. The present formulation illustrates how both fluctuations and decoherence (which results in dissipation of oscillation amplitude) and their relationship in terms of a fluctuation-dissipation–type relation in many-body physics can be realized in classical chaos without taking into consideration any phenomenological drift terms. We emphasize here that the process of dissipation of oscillation amplitude in the sense implied in this paper and in that of van Kampen [18] is related to the relaxation of correlation function but is not the dissipation of energy implied in the traditional fluctuation-dissipation theorem. We have carried out numerical computations for verification of the basic propositions which are valid for short but finite correlation time but are exact in the limit of vanishing correlation time.

The layout of the paper is as follows. In Sec. II we introduce the system–chaotic-bath model which consists of a Kubo oscillator whose stochasticity in frequency is generated from a 1.5-degree-of-freedom chaotic system (bath). Based on the theory of multiplicative noise we analyze this model in Sec. III to illustrate how diffusion and decoherence arise in classical chaos. Theoretical propositions are verified numerically in Sec. IV. The paper is concluded in Sec. V.

II. A KUBO OSCILLATOR GENERATED BY A 1.5-DEGREE-OF-FREEDOM CHAOTIC BATH

To start with we consider a harmonic oscillator with frequency ω described by a coordinate x as follows:

$$\ddot{x}+\omega^2 x=0 , \tag{1}$$

where

$$\omega^2=1+\alpha q(t)$$

is the sum of a constant part unity and a fluctuating part, $\alpha q(t)$. α is a measure of the strength of fluctuation; $q(t)$ represents the coordinate of the bath oscillator. The stochasticity in $q(t)$ is generated by the chaotic motion of a double-well oscillator driven by a time-dependent field as follows:

$$\ddot{q}+V'(q)=0 , \tag{2}$$

where

$$V(q)=aq^4-bq^2+gq\cos(\omega t) .$$

Here a, b, and g refer to the parameters of the double-well potential and the classical field, respectively.

The system-chaotic bath is thus composed of two parts. The first part described by Eq. (1) is the system of interest unperturbed part of which executes a simple harmonic motion with a constant frequency. The second part Eq. (2) plays the role of a "thermal bath." Note that in the present model we have not taken into account any feedback from the system of interest to the bath. The feedback of the system of interest to the bath is an essential requirement for realization of damping force responsible for dissipation of energy in a fluctuating system implied in the traditional fluctuation-dissipation relationship. The absence of feedback in the present model is therefore an important point of departure from the usual approach. The system of interest here acts as a probe for the noise and the noise embedded in Eq. (1) is multiplicative in nature. Thus although Eqs. (1) and (2) comprise a deterministic dynamical system, in principle, Eq. (1) may be interpreted as a stochastic differential equation with multiplicative noise [18].

Before proceeding further we would like to emphasize some relevant points at this stage. First, we consider a fully developed chaotic regime, i.e., the measure of a regular region is overwhelmingly small so that $q(t)$ may be treated as a stochastic process.

Second, in our theoretical and numerical considerations that follow we do not make any *a priori* approximation on the nature of the stochastic process $q(t)$. The special cases where $q(t)$ is a Gaussian, or a δ-function-correlated process, etc. have received so much attention in the recent literature that it is necessary to note that no such *ad hoc* approximations have been made.

Third, the strength of fluctuation α in Eq. (1) is assumed to be small, i.e., we restrict ourselves to the system-bath weak coupling regime. For calculation of relevant quantities we take care of fluctuation which is second order in α.

Fourth, the only assumption we make on the stochastic process $q(t)$ is that its correlation time is short but finite. This is a basic requirement for systematic separation of the time scales involved in the dynamics.

III. THEORETICAL CONSIDERATIONS

Over many years numerous authors [18] have studied the problem of a harmonic oscillator with fluctuating frequency in connection with wave propagation, mechanical systems, line broadening, lasers, etc. A classic comprehensive treatment has been given by van Kampen. What follows next is that we use some of its relevant standard results to the system–chaotic-bath model and show that it is possible to derive a Fokker-Planck equation whose drift and diffusion coefficients are derived explicitly from the properties of the chaotic bath. Taking into consideration the points noted in the preceding section we construct [18] the equations of motion (second order in α) for the first and second moments for the system of interest from Eq. (1). These are

$$\langle \ddot{x} \rangle+\Gamma\langle \dot{x} \rangle+\{1+\alpha c-(\alpha^2/2)c_1\}\langle x \rangle=0 \tag{3}$$

with

$$\Gamma=\tfrac{1}{2}\alpha^2 c_2$$

and

$$\frac{d}{dt}\begin{bmatrix}\langle x^2\rangle\\ \langle \dot{x}^2\rangle\\ \langle x\dot{x}\rangle\end{bmatrix}=\begin{bmatrix}0 & 0 & 2\\ \alpha^2c_3 & -\alpha^2c_2 & -2-2\alpha c\\ -1-\alpha c+\alpha c_1 & 1 & -\alpha^2c_2\end{bmatrix}\begin{bmatrix}\langle x^2\rangle\\ \langle \dot{x}^2\rangle\\ \langle x\dot{x}\rangle\end{bmatrix}, \tag{4}$$

respectively. Here c_i's are expressed as the Fourier transforms of the correlation functions of the bath as follows:

$$\begin{aligned} c_1&=\int_0^\infty \langle\langle q(t)q(t-t')\rangle\rangle \sin(2t')dt' ,\\ c_2&=\int_0^\infty \langle\langle q(t)q(t-t')\rangle\rangle (1-\cos 2t')dt' ,\\ c_3&=\int_0^\infty \langle\langle q(t)q(t-t')\rangle\rangle (1+\cos 2t')dt' ,\\ c&=\langle q(t)\rangle \text{ and } \langle\langle q_iq_j\rangle\rangle=\langle q_iq_j\rangle-\langle q_i\rangle\langle q_j\rangle . \end{aligned} \tag{5}$$

Equations (3) and (4) suggest that in principle we have effectively incorporated the properties of the chaotic bath into the evolution of the system of interest through appropriate correlation functions.

A few pertinent points are to be noted here. First, Eq. (3) reveals that chaos of the bath causes a damping of the average amplitude of the system oscillator. However, this damping term Γ may be negative, i.e., it may behave as a gain term when the fluctuations are particularly strong at twice the unperturbed frequency. The relationship

$$\Gamma=\tfrac{1}{2}\alpha^2c_2 \tag{6}$$

which connects the dissipation of amplitude (but not energy) with fluctuations is reminiscent of (not identical to) the famous fluctuation-dissipation theorem in many-body physics. It should also be noted that in addition there is a shifting term associated with the frequency of the harmonic oscillator.

Second, on examining the eigenvalues of the matrix in Eq. (4) [which up to second order in α are $-\frac{1}{4}\alpha^2(3c_2+c_3)\pm 2i\{1-\frac{1}{4}\alpha^2c_1\}$ and $\frac{1}{2}\alpha^2(c_3-c_2)$] it can be shown that zero frequency of the unperturbed case corresponding to the conservation of $\frac{1}{2}(x^2+\dot{x}^2)$ now becomes

$$\lambda_0=\tfrac{1}{2}\alpha^2(c_3-c_2)$$

or

$$\lambda_0=\tfrac{1}{2}\alpha^2\int_0^\infty \langle\langle q(t)q(t-t')\rangle\rangle\cos(2t')dt' . \tag{7}$$

This implies that owing to fluctuations in the force that have twice the characteristic frequency of the system oscillator the average energy grows at a rate λ_0 and the system becomes unstable.

Third, one obtains the dispersion in the coordinates of the system oscillator from Eqs. (3) and (4). The evolution of average amplitude is given by

$$\langle x(t)\rangle=\left[\frac{s_1x(0)+\{y(0)-rx(0)\}}{s_1-s_2}\right]e^{s_1t}+\left[\frac{s_2x(0)+\{y(0)-rx(0)\}}{s_2-s_1}\right]e^{s_2}, \tag{8}$$

where

$$\begin{aligned} r&=-\tfrac{1}{2}\alpha^2c_2 ,\\ s_1&=-\tfrac{1}{4}\alpha^2c_2+i\left[1+\alpha c-\frac{\alpha^2}{2}c_1\right]^{1/2} ,\\ s_2&=-\tfrac{1}{4}\alpha^2c_2-i\left[1+\alpha c-\frac{\alpha^2}{2}c_1\right]^{1/2} . \end{aligned}$$

Here $x(0)$ and $y(0)$ refer to average initial coordinate and momentum of the Kubo oscillator, respectively. Again solution of Eq. (4) yields

$$\langle x^2(t)\rangle=\frac{f(\lambda_0)}{(\lambda_0-\lambda_+)(\lambda_0-\lambda_-)}e^{\lambda_0t}+\frac{f(\lambda_+)}{(\lambda_+-\lambda_0)(\lambda_+-\lambda_-)}e^{\lambda_+t}+\frac{f(\lambda_-)}{(\lambda_--\lambda_0)(\lambda_--\lambda_+)}e^{\lambda_-t}, \tag{9}$$

where

$$\begin{aligned} f(\phi)&=x_1(0)(\phi+\alpha^2c_2)(\phi+\alpha^2c_2)\\ &\quad-(2-2\alpha c)x_1(0)+2x_2(0)\\ &\quad+2x_3(0)(\phi+\alpha^2c_2) \end{aligned}$$

and $x_1(0)$, $x_2(0)$, and $x_3(0)$ refer to the initial second moments as follows:

$$x_1(0)=\langle x^2\rangle,\quad x_2(0)=\langle\dot{x}^2\rangle,\quad x_3(0)=\langle x\dot{x}\rangle .$$

From Eqs. (8) and (9) we readily obtain the expression of dispersion

$$\Delta x=[\langle x^2(t)\rangle-[\langle x(t)\rangle]^2]^{1/2} .$$

It is thus apparent that an interpretation of Eq. (1) as a stochastic differential equation with multiplicative noise whose source is the chaotic bath described by Eq. (2) simply illustrates how the correlation functuation relaxes as a result of the irreversible process of decoherence and thereby implying a dissipation of oscillation amplitude and how decoherence and fluctuations are formally related through a fluctuation-dissipation–type relation. We verify numerically the basic propositions in the next section. This readily allows us to describe the stochastic evolution of the system oscillator in terms of the following Fokker-Planck equation which incorporates parametrically the properties of the chaotic bath (we follow the prescription of Ref. [18]):

$$\frac{\partial\rho}{\partial t}=-v\frac{\partial\rho}{\partial u}+u\frac{\partial\rho}{\partial v}+\alpha cu\frac{\partial\rho}{\partial v}-\alpha^2c_5\frac{\partial u\rho}{\partial v}$$
$$+\alpha^2c_4\frac{\partial^2}{\partial v^2}u^2\rho+\alpha^2c_5\frac{\partial^2}{\partial v\partial u}u^2\rho$$
$$-\alpha^2c_5\frac{\partial^2}{\partial v^2}uv\rho\ , \qquad (10)$$

where we use the notations

$$u=x,\quad v=\dot{x},\quad \langle q(t)\rangle=c\ .$$

Also note that

$$c_4=\int_0^\infty\langle\langle q(t)q(t-\tau)\rangle\rangle d\tau\ ,$$
$$c_5=\int_0^\infty\tau\langle\langle q(t)q(t-\tau)\rangle\rangle d\tau\ .$$

We now point out an essential difference between the present system–chaotic-bath model and the conventional system–N-oscillator-bath model. In the latter model the process of dissipation implies that the energy can be lost by the system of interest and absorbed by the N-oscillator bath and the conventional fluctuation-dissipation process means that the energy absorbed by the system of interest through the process of fluctuation is balanced by that absorbed by the bath through the action that the system of interest exerts on the bath. Because of the structure of coupling between the system and the chaotic bath (which implies that no energy can be transferred by the system of interest to the chaotic system) no balance of this kind is implied in the present relationship [Eq. (6)]. The Γ term [a second order term in the equation for the average; see Ref. [21] (p. 386) for a detailed discussion] in Eq. (3) is due to fluctuations and is usually dissipative (if the fluctuations at twice the frequency are not strong). This relationship between dissipation of oscillation amplitude and the autocorrelation function of fluctuation is analogous to the Green-Kubo relation in the many-body system (following van Kampen) but not identical with it because there the fluctuations are internal rather than added as a separate coupling term as in Eq. (1) itself.

To give a fair perspective of the crucial issue of dissipation (of coherence) implied in relation (6) and in the traditional fluctuation-dissipation (of energy) theorem it is pertinent to make the following two comments.

(i) The dissipation of energy is the phenomenon of waste of mechanical energy due to the interaction between the system of interest and the bath. The damping force responsible for this process implies a feedback of the system on the bath. The work of Bianucci and coworkers [16,17] is essentially devoted to a totally dynamical derivation of this process of fluctuation and dissipation by using only the dynamical properties of the chaotic systems. Dissipation takes place as a result of two key steps: the first is the action of the system of interest on the bath and the second is the regression of the bath to the invariant distribution corresponding to the given state of the system of interest. The process of fluctuation on the contrary depends on the chaotic properties of the bath.

(ii) As pointed out by van Kampen [18] there exists a second form of "dissipation" not implying absorption of energy but concerning a process of decoherence which results in a damping of the oscillation amplitude of the oscillator. This is formally related as in the standard Green-Kubo relation to the regression of equilibrium of a correlation function. While in the case mentioned by van Kampen this is caused by the action of an infinitely large number of degrees of freedom, the present work refers to the case where the process only depends on the chaotic properties of a deterministic process. Thus in a sense we realize the "dissipation" (or decoherence) process of van Kampen within the same deterministic perspective as that adopted by Bianucci and co-workers to deal with dissipation of energy.

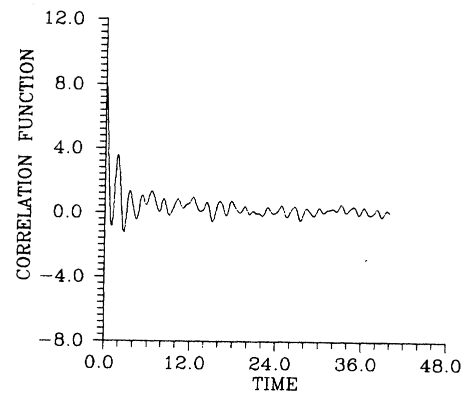

FIG. 1. Correlation function $\langle\langle q(t)q(t-\tau)\rangle\rangle$ is plotted as a function of time for parameters mentioned in the text. (Both the units are arbitrary.)

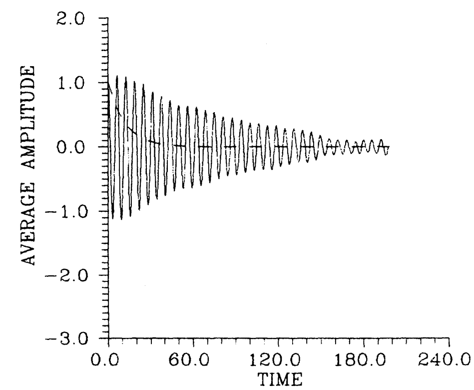

FIG. 2. Average amplitude $\langle x\rangle$ of the Kubo oscillator is plotted against time (oscillatory curve). Broken line is the theoretical curve which takes into account of the decay part of solution for $\langle x\rangle$ in Eq. (3) only. (Both the units are arbitrary.)

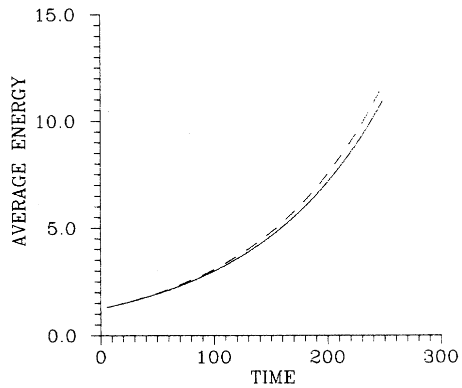

FIG. 3. Average energy of the Kubo oscillator is plotted against time (full line, numerical; broken line, theoretical). (Both the units are arbitrary.)

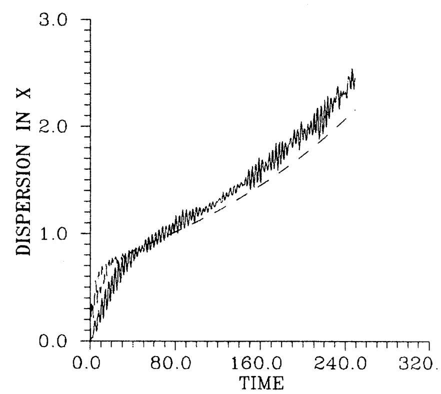

FIG. 4. Dispersion in x (coordinate of harmonic oscillator) is plotted against time (full line, numerical; broken line, theoretical). (Both the units are arbitrary.)

IV. NUMERICAL VERIFICATIONS

To verify the basic relations in Eqs. (6) and (7) and the relation for dispersion we have first solved Eqs. (1) and (2) by the fourth order Runge-Kutta method. The parameters chosen [19] are as follows: $a=0.5$, $b=10.0$, and $\omega=6.07$. The driving field amplitude g is set at 10 to achieve well-developed global chaos. The strength of fluctuation α is assumed to be 0.1. To calculate the average quantities from Eqs. (1) and (2) the averaging is carried out over 1000 trajectories for a given set. A typical plot of decay of correlation function is shown in Fig. 1. This decay is a well-known characteristic signature of classical chaos [20,22].

In Fig. 2 we show how the average amplitude $\langle x \rangle$ of oscillation of the system oscillator decays with time. The rate of damping of the oscillations is found to be in excellent agreement with that calculated theoretically (broken line) from Eq. (6) by making use of the correlation function and its Fourier transform. [Note that in the theoretical curve we have not taken into consideration the oscillatory terms in Eq. (3).]

In Fig. 3 we show how the average energy of the system oscillator calculated by direct numerical integration of Eqs. (1) and (2) and by subsequent trajectory averaging diverges in time. The result is compared with that calculated theoretically (broken line) using the relation (7) with the help of correlation function and its transform.

Figure 4 demonstrates a relative comparison between a numerical and a theoretical plot (dotted) for dispersion of coordinates of the system oscillator. It is apparent that in the numerical curve the irregular oscillation persists for a long time where the oscillations are smoothed out in time in the theoretical curve. The rate of divergence in the two cases are in very good agreement. It is apparent that because of multiplicative noise with no feedback the diffusive motion of the system oscillator driven by a chaotic bath becomes strongly nonlinear.

To check the validity of relations (6) and (7) and the relation for dispersion we have carried out further numerical computation by varying g. The plots qualitatively remain the same. For the sake of brevity we have not reproduced them here as a part of our discussion. Keeping in mind the fact that our theoretical calculation is correct up to second order (although it can be extended to higher order) in α the order of magnitude agreement demonstrates the qualitative validity of the proposed relationships in classical chaos.

V. CONCLUSIONS

The key point in the deterministic derivation of a stochastic process on the basis of traditional system–N-oscillator heat-bath model rests on appropriate elimination of bath degrees of freedom and incorporation of the effect of the bath on the evolution of the relevant part of the system through bath correlation functions. It is in this spirit the present approach illustrates how fluctuation and decoherence (and their relationship in terms of a Green-Kubo type relation) can be realized as fundamental process of a chaotic dynamics through a system–chaotic-bath model (where the bath is a 1.5-degree-of-freedom chaotic system) by considering the evolution of the system of interest which takes into account the effect of chaos through appropriate bath correlation functions. Although in the majority of the theoretical treatments diffusion appears naturally, drift terms do have some phenomenological bearing. However, both of these processes actively participate in a multiplicative process as considered in the present case in such a way that both of them can be treated on a common theoretical footing.

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